

# UNIMODALITY ON $\delta$ -VECTORS OF LATTICE POLYTOPES AND TWO RELATED PROPERTIES

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**ABSTRACT.** In this paper, we investigate two properties concerning the unimodality of the  $\delta$ -vectors of lattice polytopes, which are log-concavity and alternatingly increasingness. For lattice polytopes  $\mathcal{P}$  of dimension  $d$ , we prove that the dilated lattice polytopes  $n\mathcal{P}$  have strictly log-concave and strictly alternatingly increasing  $\delta$ -vectors if  $n > \max\{s, d+1-s\}$ , where  $s$  is the degree of the  $\delta$ -polynomial of  $\mathcal{P}$ . The bound  $\max\{s, d+1-s\}$  for  $n$  is reasonable. We also provide several kinds of unimodal (or non-unimodal)  $\delta$ -vectors. Concretely, we give examples of lattice polytopes whose  $\delta$ -vectors are not unimodal, unimodal but neither log-concave nor alternatingly increasing, alternatingly increasing but not log-concave, and log-concave but not alternatingly increasing, respectively.

## 1. INTRODUCTION

The  $\delta$ -vectors of lattice polytopes are one of the most fascinating objects on enumerative combinatorics. In this paper, we focus on the unimodality question on  $\delta$ -vectors of lattice polytopes and investigate two related properties on the unimodality, called “log-concave” and “alternatingly increasing”.

Let  $\mathcal{P} \subset \mathbb{R}^N$  be a *lattice* polytope of dimension  $d$ , which is a convex polytope all of whose vertices are lattice points in the lattice  $\mathbb{Z}^N$ . Given a positive integer  $m$ , we define

$$i(\mathcal{P}, m) = |m\mathcal{P} \cap \mathbb{Z}^N|,$$

where  $m\mathcal{P} = \{m\alpha : \alpha \in \mathcal{P}\}$  and  $|\cdot|$  denotes the cardinality. The enumerative function  $i(\mathcal{P}, m)$  is actually a polynomial in  $m$  of degree  $d$  with its constant term 1 ([7]). This polynomial  $i(\mathcal{P}, m)$  is called the *Ehrhart polynomial* of  $\mathcal{P}$ . Moreover,  $i(\mathcal{P}, m)$  satisfies *Ehrhart–Macdonald reciprocity* (see [2, Theorem 4.1]):

$$(1.1) \quad |m\mathcal{P}^\circ \cap \mathbb{Z}^N| = (-1)^d i(\mathcal{P}, -m) \text{ for each integer } m > 0,$$

where  $\mathcal{P}^\circ$  denotes the relative interior of  $\mathcal{P}$ . We refer the reader to [2, Chapter 3] or [8, Part II] for the introduction to the theory of Ehrhart polynomials.

We define the sequence  $\delta_0, \delta_1, \dots, \delta_d$  of integers by the formula

$$(1-t)^{d+1} \left( 1 + \sum_{m=1}^{\infty} i(\mathcal{P}, m)t^m \right) = \sum_{i=0}^d \delta_i t^i.$$

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We call the integer sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$$

the  $\delta$ -vector (also called *Ehrhart  $\delta$ -vector* or  *$h^*$ -vector*) of  $\mathcal{P}$  and the polynomial  $\delta_{\mathcal{P}}(t) = \delta_0 + \delta_1 t + \dots + \delta_d t^d$  the  $\delta$ -polynomial of  $\mathcal{P}$ .

By (1.1), one has

$$(1-t)^{d+1} \sum_{m=1}^{\infty} |m\mathcal{P}^\circ \cap \mathbb{Z}^N| t^m = \sum_{i=0}^d \delta_{d-i} t^{i+1}.$$

Thus it follows that

$$(1.2) \quad \min\{k : k\mathcal{P}^\circ \cap \mathbb{Z}^N \neq \emptyset\} = d+1 - \max\{i : \delta_i \neq 0\}.$$

The  $\delta$ -vectors of lattice polytopes have the following properties:

- $\delta_0 = 1$ ,  $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$  and  $\delta_d = |\mathcal{P}^\circ \cap \mathbb{Z}^N|$ . Hence,  $\delta_1 \geq \delta_d$ . In particular, when  $\delta_1 = \delta_d$ ,  $\mathcal{P}$  must be a simplex.
- Each  $\delta_i$  is nonnegative ([14]).
- If  $\mathcal{P}^\circ \cap \mathbb{Z}^N$  is nonempty, i.e.,  $\delta_d > 0$ , then  $\delta_1 \leq \delta_i$  for each  $1 \leq i \leq d-1$  ([10]).
- The leading coefficient  $(\sum_{i=0}^d \delta_i)/d!$  of  $i(\mathcal{P}, n)$  is equal to the volume of  $\mathcal{P}$  ([2, Corollary 3.20, 3.21]).

There are two well-known inequalities on  $\delta$ -vectors. Let  $s = \max\{i : \delta_i \neq 0\}$ . One is

$$(1.3) \quad \delta_0 + \delta_1 + \dots + \delta_i \leq \delta_s + \delta_{s-1} + \dots + \delta_{s-i}, \quad 0 \leq i \leq s,$$

which is proved by Stanley [16], and another one is

$$(1.4) \quad \delta_d + \delta_{d-1} + \dots + \delta_{d-i} \leq \delta_1 + \delta_2 + \dots + \delta_{i+1}, \quad 0 \leq i \leq d-1,$$

which appears in the work of Hibi [10, Remark (1.4)].

For a lattice polytope  $\mathcal{P} \subset \mathbb{R}^N$ , we say that  $\mathcal{P}$  has the *integer decomposition property* (IDP, for short) if for each integer  $\ell \geq 1$  and  $\alpha \in \ell\mathcal{P} \cap \mathbb{Z}^N$ , there are  $\alpha_1, \dots, \alpha_\ell$  in  $\mathcal{P} \cap \mathbb{Z}^N$  such that  $\alpha = \alpha_1 + \dots + \alpha_\ell$ . Having IDP is also known as what is *integrally closed*.

We also recall the following three notions. Let  $(a_0, a_1, \dots, a_d)$  be a sequence of real numbers.

- We say that  $(a_0, a_1, \dots, a_d)$  is *unimodal* if there is some  $c$  with  $0 \leq c \leq d$  such that

$$a_0 \leq a_1 \leq \dots \leq a_c \geq a_{c+1} \geq \dots \geq a_d.$$

If each inequality is strict, then we say that it is *strictly unimodal*.

- $(a_0, a_1, \dots, a_d)$  is called *log-concave* if for each  $1 \leq i \leq d-1$ , one has

$$a_i^2 \geq a_{i-1}a_{i+1}.$$

If  $a_i^2 > a_{i-1}a_{i+1}$  for each  $i$ , then it is called *strictly log-concave*.

- ([13, Definition 2.9]) We call  $(a_0, a_1, \dots, a_d)$  *alternatingly increasing* if  $a_i \leq a_{d-i}$  for  $0 \leq i \leq \lfloor (d-1)/2 \rfloor$  and  $a_{d+1-i} \leq a_i$  for  $1 \leq i \leq \lfloor d/2 \rfloor$  are satisfied. Namely,

$$a_0 \leq a_d \leq a_1 \leq a_{d-1} \leq \dots \leq a_{\lfloor (d-1)/2 \rfloor} \leq a_{d-\lfloor (d-1)/2 \rfloor} \leq a_{\lfloor (d+1)/2 \rfloor}.$$

If each inequality is strict, then we call it *strictly alternatingly increasing*.

Note that  $(a_0, \dots, a_d)$  is unimodal (resp. strictly unimodal) if it is log-concave (resp. log-concave) or alternatingly increasing (resp. strictly alternatingly increasing).

Our motivation to organize this paper is to give some answer for the following:

**Question 1.1.** *Let  $\mathcal{P}$  be a lattice polytope having IDP with at least one interior lattice point. Then is  $\delta(\mathcal{P})$  always unimodal?*

The similar question is also mentioned in [13, Question 1.1]. Moreover, the following has been conjectured by Stanley [15] in 1989: the  $h$ -vectors of standard graded Cohen–Macaulay domains are always unimodal. This conjecture still seems to be open. We note that Question 1.1 is the case of Ehrhart rings (see [8, Part II]) for this question with additional condition “ $a$ -invariant  $-1$ ”.

For this question, the following facts on the unimodality of the  $\delta$ -vectors of lattice polytopes are known:

- (1) If  $\mathcal{P} \subset \mathbb{R}^d$  is a *reflexive polytope* (introduced in [1]), which is a lattice polytope whose dual polytope  $\mathcal{P}^\vee = \{y \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } x \in \mathcal{P}\}$  is also a lattice polytope, of dimension at most 5, then  $\delta(\mathcal{P})$  is unimodal. This follows from [10, Theorem 1.1] and [9]. Note that every reflexive polytope contains exactly one interior lattice point.
- (2) Hibi conjectured that all the  $\delta$ -vectors of reflexive polytopes are unimodal ([8, §36]). However, counterexamples were found by Mustařă and Payne [11, 12]. On the other hand, their counterexamples do not have IDP. It may be still open whether there exists a reflexive polytope having IDP whose  $\delta$ -vector is not unimodal.
- (3) Bruns and R  mer [5] proved that each reflexive polytope with a regular unimodular triangulation has a unimodal  $\delta$ -vector. Note that if a lattice polytope has a regular unimodular triangulation, then it also has IDP, while the converse is not true in general.
- (4) Schepers and Van Langenhoven [13, Proposition 2.17] proved that every parallelepiped with at least one interior lattice point has an alternatingly increasing  $\delta$ -vector.

In this paper, as a further contribution for Question 1.1, we prove the following:

**Theorem 1.2.** *Let  $\mathcal{P} \subset \mathbb{R}^N$  be a lattice polytope of dimension  $d$  and  $s$  the degree of  $\delta_{\mathcal{P}}(t)$ . Let  $(\delta_0, \delta_1, \dots, \delta_d)$  be the  $\delta$ -vector of the dilated polytope  $n\mathcal{P}$  for a positive integer  $n$ . Then the following statements hold:*

- (i)  $(\delta_0, \delta_1, \dots, \delta_d)$  is strictly log-concave when  $n \geq s$ ;
- (ii) We have  $\delta_i \leq \delta_{d-i}$  for  $1 \leq i \leq \lfloor (d-1)/2 \rfloor$  and  $\delta_{d+1-i} < \delta_i$  for  $1 \leq i \leq \lfloor d/2 \rfloor$  when  $n \geq \max\{s, d+1-s\}$ . Moreover, if  $|(d+1-s)\mathcal{P}^\circ \cap \mathbb{Z}^N| > 1$ , then we have  $\delta_i < \delta_{d-i}$  for  $0 \leq i \leq \lfloor (d-1)/2 \rfloor$ . Hence,  $(\delta_0, \delta_1, \dots, \delta_d)$  is strictly alternatingly increasing when  $n \geq \max\{s, d+1-s\}$  with  $|(d+1-s)\mathcal{P}^\circ \cap \mathbb{Z}^N| > 1$ , or when  $n > \max\{s, d+1-s\}$ .

In [3] and [4], it has been proved that for a lattice polytope  $\mathcal{P}$  of dimension  $d$ , there exists an integer  $n_d$  such that the  $\delta$ -vector of a dilated polytope  $n\mathcal{P}$  are strictly log-concave and strictly alternatingly increasing for each  $n \geq n_d$ . Theorem 1.2 gives an explicit bound

for  $n_d$ . Moreover, the following remark says that our bound  $\max\{s, d+1-s\}$  is reasonable in some sense.

**Remark 1.3.** By (1.2), we see that  $n\mathcal{P}^\circ \cap \mathbb{Z}^N \neq \emptyset$  if and only if  $n \geq d+1-s$ .

Moreover, by [6, Theorem 1.1], the inequality  $\mu_{\text{idp}}(\mathcal{P}) \leq \mu_{\text{Ehr}}(\mathcal{P})$  holds. Since  $\mu_{\text{Ehr}}(\mathcal{P}) = \max\{i : \delta_i \neq 0\} = s$ , we see that  $n\mathcal{P}$  has IDP if  $n \geq s$ . In addition, when  $\mu_{\text{idp}}(\mathcal{P}) = \mu_{\text{Ehr}}(\mathcal{P})$ , this bound is sharp.

Therefore,  $n\mathcal{P}$  has IDP and contains at least one interior lattice point if  $n \geq \max\{s, d+1-s\}$  and the bound  $\max\{s, d+1-s\}$  sometimes becomes optimal.

For the proof of Theorem 1.2, we prove a more general statement (Theorem 2.1) in Section 2. (See Remark 2.2, too.)

Moreover, we also provide several kinds of  $\delta$ -vectors of lattice polytopes concerning the unimodality of  $\delta$ -vectors. We construct an infinite family of lattice polytopes whose  $\delta$ -vectors are not unimodal in Section 3.1, unimodal but neither log-concave nor alternatingly increasing for even dimensions in Section 3.2, alternatingly increasing but not log-concave in Section 3.3, and log-concave but not alternatingly increasing for low dimensions, respectively. (See Figure 1.)

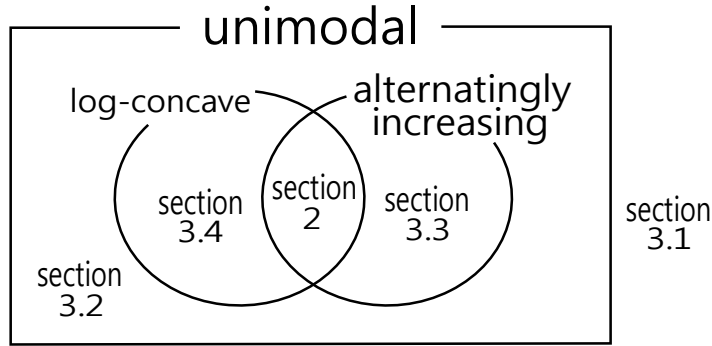


FIGURE 1. Organization of this paper

## 2. UNIMODALITY ON $\delta$ -VECTORS OF DILATED POLYTOPES

We recall some notation from [3]. For a polynomial  $h(t) = h_0 + h_1t + \cdots + h_d t^d$  in  $t$  of degree at most  $d$  with nonnegative coefficients and  $h_0 = 1$ , there is a polynomial  $g(m)$  in  $m$  of degree  $d$  such that

$$\sum_{m=0}^{\infty} g(m)t^m = \frac{h(t)}{(1-t)^{d+1}}.$$

(See [2, Lemma 3.9].) Note that the polynomial  $g(m)$  can be written like  $g(m) = \sum_{i=0}^d h_i \binom{m+d-i}{d}$ . For each integer  $n$ , we define  $U_n h(t) = h_0^{(n)} + h_1^{(n)}t + \cdots + h_d^{(n)}t^d$  as follows:

$$\sum_{m=0}^{\infty} g(nm)t^m = \frac{U_n h(t)}{(1-t)^{d+1}}.$$

The main result of this paper is the following:

**Theorem 2.1.** Let  $d \geq 5$  and let  $h(t) = \sum_{j=0}^d h_j t^j$  be a polynomial in  $t$  of degree  $s$ , where  $s \leq d$ , with nonnegative coefficients and  $h_0 = 1$ . Let  $n$  be a positive integer and  $\delta_i = h_i^{(n)}$  the coefficient of  $U_n h(t)$  for  $0 \leq i \leq d$ . Then the following statements hold:

- (i)  $(\delta_0, \delta_1, \dots, \delta_d)$  is strictly log-concave when  $n \geq s$ ;
- (ii) If  $h_0, h_1, \dots, h_d$  satisfy the inequalities

$$(2.1) \quad h_0 + h_1 + \dots + h_i \leq h_s + h_{s-1} + \dots + h_{s-i} \quad \text{for } 0 \leq i \leq \lfloor s/2 \rfloor$$

and

$$(2.2) \quad h_d + h_{d-1} + \dots + h_{d-i} \leq h_1 + h_2 + \dots + h_{i+1} \quad \text{for } 0 \leq i \leq \lfloor (d+1)/2 \rfloor,$$

then we have  $\delta_j \leq \delta_{d-j}$  for  $1 \leq j \leq \lfloor (d-1)/2 \rfloor$  and  $\delta_{d+1-k} < \delta_k$  for  $1 \leq k \leq \lfloor d/2 \rfloor$  when  $n \geq \max\{s, d+1-s\}$ . Moreover, if  $h_s > 1$ , then we also have  $\delta_j < \delta_{d-j}$  for  $0 \leq j \leq \lfloor (d-1)/2 \rfloor$ , and thus,  $(\delta_0, \delta_1, \dots, \delta_d)$  is strictly alternatingly increasing.

**Remark 2.2.** Since the inequalities (2.1) and (2.2) are nothing but the inequalities (1.3) and (1.4) for  $\delta$ -vectors, the  $\delta$ -vectors of lattice polytopes satisfy all the conditions in Theorem 2.1 for  $h_0, h_1, \dots, h_d$ . Therefore, as an immediate corollary of Theorem 2.1, we obtain Theorem 1.2.

For the proof of Theorem 2.1, we recall a useful lemma for the computation of  $U_n h(t)$ .

**Lemma 2.3** (cf. [3, Lemma 3.2]). Let  $h(t) = \sum_{j=0}^d h_j t^j$  be the same as in Theorem 2.1 and let  $U_n h(t) = \sum_{j=0}^d \delta_j t^j$ . Then one has  $\delta_i = c_{ni}$  for each  $0 \leq i \leq d$ , where  $c_j$  is the coefficient of the polynomial  $h(t)(1+t+\dots+t^{n-1})^{d+1}$  in  $t$ .

Moreover, we also recall the following fundamental assertion for log-concave sequences.

**Lemma 2.4.** Let  $b_1, b_2, \dots$  be a (resp. strictly) log-concave sequence of nonnegative real numbers such that  $b_k = b_{k+1} = \dots = 0$  if  $b_k = 0$  for some  $k$ . Then we have  $b_i b_j \geq b_{i-m} b_{j+m}$  (resp.  $b_i b_j > b_{i-m} b_{j+m}$ ) for any  $i \leq j$  and  $m \geq 0$ .

*Proof.* It suffices to show that  $b_i b_j \geq b_{i-1} b_{j+1}$  for every  $i \leq j$ . When  $b_i = 0$  or  $b_j = 0$ , one has  $b_{j+1} = 0$ . Thus the equality holds.

Assume that  $b_i > 0$  and  $b_j > 0$ . Thus, in particular, we have  $b_k > 0$  for each  $i \leq k \leq j$ . Since  $b_i^2 \geq b_{i-1} b_{i+1}$  and  $b_i > 0$ , one has  $b_i \geq b_{i-1} b_{i+1} / b_i$ . Similarly, since  $b_{i+1} \geq b_i b_{i+2} / b_{i+1}$ , we obtain  $b_i \geq b_{i-1} / b_i \cdot b_i b_{i+2} / b_{i+1} = b_{i-1} b_{i+2} / b_{i+1}$ . By repeating this computation, we obtain

$$b_i \geq b_{i-1} b_{i+1} / b_i \geq b_{i-1} b_{i+2} / b_{i+1} \geq \dots \geq b_{i-1} b_{j+1} / b_j.$$

Hence,  $b_i b_j \geq b_{i-1} b_{j+1}$  holds. The case of strictly log-concave sequences is similar.  $\square$

For a sequence of numbers  $b_0, b_1, \dots$ , let  $I(b_n)$  denote the index of  $b_n$ , i.e.,  $I(b_n) = n$ .

*Proof of Theorem 2.1.* Our goal is to show that  $\delta_0, \delta_1, \dots, \delta_d$  satisfy the following inequalities:

- (i)  $\delta_i^2 > \delta_{i-1} \delta_{i+1}$  for  $1 \leq i \leq d-1$ ;
- (ii-a)  $\delta_i \leq \delta_{d-i}$  for  $1 \leq i \leq \lfloor (d-1)/2 \rfloor$  and  $\delta_i < \delta_{d-i}$  for  $0 \leq i \leq \lfloor (d-1)/2 \rfloor$  if  $h_s > 1$ ;
- (ii-b)  $\delta_{d+1-i} < \delta_i$  for  $1 \leq i \leq \lfloor d/2 \rfloor$ .

Let  $a_0, a_1, \dots, a_{(d+1)(n-1)}$  be the integers such that  $\sum_{i=0}^{(d+1)(n-1)} a_i t^i = (1+t+\dots+t^{n-1})^{d+1}$ . By Lemma 2.3, we have

$$\delta_0 = 1 \text{ and } \delta_i = \sum_{j=0}^s h_j a_{ni-j} \text{ for } 1 \leq i \leq d,$$

where we let  $a_k = 0$  if  $k > (d+1)(n-1)$ .

Since the coefficients of the polynomial  $(1+t+\dots+t^{n-1})^2$  are symmetric and strictly log-concave, so are the coefficients of the polynomial  $\sum_{i=0}^{(d+1)(n-1)} a_i t^i = (1+t+\dots+t^{n-1})^{d+1}$ . (See [15, Proposition 1 and Proposition 2].) In particular, we have

$$(2.3) \quad a_i = a_{(d+1)(n-1)-i} \text{ for } 0 \leq i \leq (d+1)(n-1),$$

$$(2.4) \quad a_i^2 > a_{i-1} a_{i+1} \text{ for } 1 \leq i \leq (d+1)(n-1) - 1, \text{ and}$$

$$(2.5) \quad a_i > a_{i-1} \text{ for } 1 \leq i \leq \lfloor (d+1)(n-1)/2 \rfloor.$$

(i) We can compute as follows:

$$\begin{aligned} & \delta_i^2 - \delta_{i-1} \delta_{i+1} \\ &= \left( \sum_{j=0}^s h_j a_{ni-j} \right)^2 - \left( \sum_{j=0}^s h_j a_{n(i-1)-j} \right) \left( \sum_{j=0}^s h_j a_{n(i+1)-j} \right) \\ &= \sum_{j=0}^s h_j^2 (a_{ni-j}^2 - a_{n(i-1)-j} a_{n(i+1)-j}) \\ &+ \sum_{0 \leq p < q \leq s} h_p h_q ((a_{ni-p} a_{ni-q} - a_{n(i-1)-p} a_{n(i+1)-q}) + (a_{ni-p} a_{ni-q} - a_{n(i-1)-q} a_{n(i+1)-p})). \end{aligned}$$

By (2.4) and Lemma 2.4, we immediately obtain

$$a_{ni-j}^2 - a_{ni-j-n} a_{ni-j+n} > 0 \text{ and } a_{ni-q} a_{ni-p} - a_{ni-q-n} a_{ni-p+n} > 0.$$

Moreover, since  $n \geq s$ , we have  $I(a_{ni-q}) - I(a_{n(i-1)-p}) = I(a_{n(i+1)-q}) - I(a_{ni-p}) = n - q + p \geq n - s \geq 0$ . Thus we also obtain

$$a_{ni-q} a_{ni-p} - a_{n(i-1)-p} a_{n(i+1)-q} > 0$$

by (2.4) and Lemma 2.4. From the nonnegativity of  $h_i$  together with  $h_0 = 1$ , we conclude that  $\delta_i^2 - \delta_{i-1} \delta_{i+1} \geq h_0^2 (a_{ni}^2 - a_{n(i-1)} a_{n(i+1)}) > 0$ , as required.

(ii-a) Fix  $1 \leq i \leq \lfloor (d-1)/2 \rfloor$ . In the sequel, we will prove  $\delta_{d-i} - \delta_i \geq 0$  and the strictly one holds if  $h_s > 1$ . Note that  $\delta_d \geq h_s > 1 = \delta_0$  for  $n \geq \max\{s, d+1-s\}$ .

Let  $k_0, \dots, k_\ell$  be the indices such that  $\sum_{r=0}^d h_r t^r = h_{k_0} + h_{k_1} t^{k_1} + \dots + h_{k_\ell} t^{k_\ell}$ , where  $0 = k_0 < k_1 < \dots < k_\ell = s$  and  $h_{k_j} > 0$  for each  $0 \leq j \leq \ell$ . For  $0 \leq j < \ell$ , if  $h_{k_0} + \dots + h_{k_j} \leq h_{k_\ell} = h_s$ , then we set  $m(j) = \ell$ ; otherwise let  $m(j)$  be a unique integer with  $0 \leq m(j) < \ell$  such that

$$h_{k_\ell} + \dots + h_{k_{m(j)+1}} < h_{k_0} + \dots + h_{k_j} \leq h_{k_\ell} + \dots + h_{k_{m(j)}}.$$

Clearly,  $m(j-1) \geq m(j)$ . Moreover, we have

$$(2.6) \quad k_j + k_{m(j)} \geq s.$$

In fact,  $h_{k_\ell} + \cdots + h_{k_\ell - k_j} \geq h_{k_0} + \cdots + h_{k_j}$  should be satisfied by (2.1), while  $h_{k_\ell} + \cdots + h_{k_{m(j)+1}} = h_{k_\ell} + \cdots + h_{k_{m(j)+1}} < h_{k_0} + \cdots + h_{k_j}$  holds. Thus, we have  $k_{m(j)} + 1 > k_\ell - k_j$ , i.e.,  $k_j + k_{m(j)} \geq k_\ell = s$ .

Let  $t = \max\{j : j \leq m(j)\}$ . In particular, we have

$$0 = k_0 < k_1 < \cdots < k_t \leq k_{m(t)} \leq k_{m(t-1)} \leq \cdots \leq k_{m(0)} = k_\ell.$$

Then we see that  $m(t) = t$  or  $m(t) = t + 1$ . In fact, on the contrary, suppose that  $m(t) \geq t + 2$ . Since  $m(t + 1) < t + 1$ , we have

$$\begin{aligned} h_{k_0} + \cdots + h_{k_t} + h_{k_{t+1}} &> h_{k_\ell} + \cdots + h_{k_{m(t)}} + \cdots + h_{k_{t+2}} + h_{k_{t+1}} + \cdots + h_{k_{m(t+1)+1}} \\ &\geq h_{k_\ell} + \cdots + h_{k_{m(t)}} + h_{k_{t+1}} \geq h_{k_0} + \cdots + h_{k_t} + h_{k_{t+1}}, \end{aligned}$$

a contradiction.

For each  $0 \leq p, q \leq \ell$ , let  $A(p, q) = (a_{n(d-i)-k_p} - a_{ni-k_q}) + (a_{n(d-i)-k_q} - a_{ni-k_p})$ . Then we have  $A(j, m(j)) \geq 0$  for  $0 \leq j \leq \ell$ . In fact, by (2.3), we have  $a_{n(d-i)-u} = a_{n(i+1)-d-1+u}$ . Thus

$$\begin{aligned} I(a_{n(i+1)-d-1+k_j}) - I(a_{ni-k_{m(j)}}) &= I(a_{n(i+1)-d-1+k_{m(j)}}) - I(a_{ni-k_j}) \\ &= n - d - 1 + k_j + k_{m(j)} \geq n - (d + 1 - s) \geq 0 \end{aligned}$$

by (2.6). When  $1 \leq i \leq (d - 2)/2$ , one has

$$\begin{aligned} &\lfloor (d + 1)(n - 1)/2 \rfloor - \max\{I(a_{n(i+1)-d-1+k_j}), I(a_{n(i+1)-d-1+k_{m(j)}})\} \\ &\geq ((d + 1)(n - 1) - 1)/2 - (nd/2 - d - 1) - \max\{k_j, k_{m(j)}\} \geq (n + d - 2s)/2 \geq 0. \end{aligned}$$

When  $i = (d - 1)/2$ , if  $k_j \geq (d + 1)/2$ , then one has

$$\lfloor (d + 1)(n - 1)/2 \rfloor - I(a_{n(d+1)/2-k_j}) \geq k_j - (d + 1)/2 \geq 0,$$

and if  $k_j \leq (d + 1)/2$ , since  $a_{n(d+1)/2-k_j} = a_{(n-2)(d+1)/2+k_j}$  by (2.3), one also has

$$\lfloor (d + 1)(n - 1)/2 \rfloor - I(a_{(n-2)(d+1)/2+k_j}) \geq (d + 1)/2 - k_j \geq 0.$$

Similarly, one also has

$$\lfloor (d + 1)(n - 1)/2 \rfloor \geq I(a_{n(d+1)/2-k_{m(j)}}) \text{ or } \lfloor (d + 1)(n - 1)/2 \rfloor \geq I(a_{(n-2)(d+1)/2+k_{m(j)}}).$$

Therefore, by (2.5), we conclude that  $A(j, m(j)) \geq 0$  for each  $0 \leq j \leq t$ . In the same way, we also conclude that  $A(j, r) \geq 0$  for each  $m(j) \leq r \leq m(j - 1)$ .

Moreover, for  $1 \leq j \leq m(t)$ , we define  $f_j$  by setting

$$f_j = \begin{cases} \left( \sum_{r=m(j-1)}^{\ell} h_{k_r} - \sum_{r=0}^{j-1} h_{k_r} \right) A(j, m(j-1)) + \sum_{r=m(j)+1}^{m(j-1)-1} h_{k_r} A(j, r) \\ \quad + \left( \sum_{r=0}^j h_{k_r} - \sum_{r=m(j)+1}^{\ell} h_{k_r} \right) A(j, m(j)), & \text{if } m(j) < m(j-1), \\ h_{k_j} A(j, m(j)), & \text{if } m(j) = m(j-1) \end{cases}$$

for  $1 \leq j \leq m(t) - 1$ , and

$$f_t = \begin{cases} \left( \sum_{r=m(t-1)}^{\ell} h_{k_r} - \sum_{r=0}^{t-1} h_{k_r} \right) A(t, m(t-1)) + \sum_{r=m(t)+1}^{m(t-1)-1} h_{k_r} A(t, r) \\ \quad + \left( \sum_{r=0}^t h_{k_r} - \sum_{r=m(t)+1}^{\ell} h_{k_r} \right) A(t, t)/2, & \text{if } m(t) < m(t-1), \\ \left( \sum_{r=m(t-1)}^{\ell} h_{k_r} - \sum_{r=0}^{t-1} h_{k_r} \right) A(t, t)/2, & \text{if } m(t) = m(t-1) \end{cases}$$

when  $m(t) = t$  and

$$f_{t+1} = \left( \sum_{r=m(t)}^{\ell} h_{k_r} - \sum_{r=0}^t h_{k_r} \right) A(t+1, t+1)/2$$

when  $m(t) = t+1$ . By definition of  $m(j)$  together with the nonnegativity of each  $A(j, r)$  for  $m(j) \leq r \leq m(j-1)$ , we obtain that  $f_j \geq 0$  for each  $j$ .

By using these notation, we can compute as follows:

$$\delta_{d-i} - \delta_i = \sum_{j=0}^{\ell} h_{k_j} a_{n(d-i)-k_j} - \sum_{j=0}^{\ell} h_{k_j} a_{ni-k_j} = A(0, \ell) + \sum_{j=1}^{m(t)} f_j \geq 0.$$

Furthermore, if  $h_s > 1$ , then we have

$$f_1 \geq \left( \sum_{r=m(0)}^{\ell} h_{k_r} - h_{k_0} \right) A(1, m(0)) \geq h_s - 1 > 0 \text{ when } m(1) < m(0) = \ell,$$

$$f_1 = h_{k_1} A(1, m(1)) = h_{k_1} A(1, m(0)) \geq h_{k_1} > 0 \text{ when } m(1) = m(0).$$

Therefore, we also obtain that  $\delta_{d-i} > \delta_i$  if  $h_s > 1$ .

(ii-b) Fix  $1 \leq i \leq \lfloor d/2 \rfloor$ . In the sequel, we will prove  $\delta_i - \delta_{d+1-i} > 0$ . We employ the similar technique to the above (ii-a).

Let  $k_0, \dots, k_{\ell}$  be the same things as above. For  $0 < j \leq \ell$ , if  $h_{k_1} \geq h_{k_{\ell}} + \dots + h_{k_j}$ , then we set  $n(j) = 1$ ; otherwise let  $n(j)$  be a unique integer with  $1 < n(j) \leq \ell$  such that

$$h_{k_1} + \dots + h_{k_{n(j)-1}} < h_{k_{\ell}} + \dots + h_{k_j} \leq h_{k_1} + \dots + h_{k_{n(j)}}.$$

Clearly,  $n(j-1) \geq n(j)$ . Moreover, we have

$$(2.7) \quad k_j + k_{n(j)} \leq d+1.$$

In fact,

$$h_{k_1} + \dots + h_{d+1-k_j} = h_1 + \dots + h_{d+1-k_j} \geq h_d + \dots + h_{k_j} = h_{k_{\ell}} + \dots + h_{k_j}$$

should be satisfied by (2.2), while  $h_{k_1} + \dots + h_{k_{n(j)-1}} = h_1 + \dots + h_{k_{n(j)-1}} < h_{k_{\ell}} + \dots + h_{k_j}$  holds. Thus, we have  $k_{n(j)} - 1 < d+1 - k_j$ , i.e.,  $k_j + k_{n(j)} \leq d+1$ .

Let  $t' = \min\{j : j \geq n(j)\}$ . In particular, we have

$$1 \leq k_{n(\ell)} \leq k_{n(\ell-1)} \leq \dots \leq k_{n(t')} \leq k_{t'} < k_{t'+1} < \dots < k_{\ell}.$$

Then  $n(t') = t'$  or  $n(t') = t' - 1$ . In fact, on the contrary, suppose that  $n(t') \leq t' - 2$ . Since  $n(t' - 1) > t' - 1$ , we have

$$\begin{aligned} h_{k_{\ell}} + \dots + h_{k_{t'-1}} &> h_{k_1} + \dots + h_{k_{n(t')}} + \dots + h_{k_{t'-2}} + h_{k_{t'-1}} + \dots + h_{k_{n(t'-1)-1}} \\ &\geq h_{k_1} + \dots + h_{k_{n(t')}} + h_{k_{t'-1}} \geq h_{k_{\ell}} + \dots + h_{k_{t'}} + h_{k_{t'-1}}, \end{aligned}$$

a contradiction.

For each  $0 \leq p, q \leq \ell$ , let  $B(p, q) = (a_{ni-k_p} - a_{n(d+1-i)-k_q}) + (a_{ni-k_q} - a_{n(d+1-i)-k_p})$ . Then we have  $B(j, n(j)) \geq 0$ . In fact, by (2.3), we have  $a_{n(d+1-i)-u} = a_{ni-d-1+u}$ . Thus

$$I(a_{ni-k_j}) - I(a_{ni-d-1+k_{n(j)}}) = I(a_{ni-k_{n(j)}}) - I(a_{ni-d-1+k_j}) = d+1 - (k_j + k_{n(j)}) \geq 0$$



by (2.7). When  $1 \leq i \leq (d-1)/2$ , one has

$$\begin{aligned} & \lfloor (d+1)(n-1)/2 \rfloor - \max\{I(a_{ni-k_j}), I(a_{ni-k_{n(j)}})\} \\ & \geq ((d+1)(n-1)-1)/2 - n(d-1)/2 + \min\{k_j, k_{n(j)}\} \geq n - d/2 > 0 \end{aligned}$$

by  $n \geq (d+1)/2$ . When  $i = d/2$ , if  $n/2 + k_j \geq d/2 + 1$ , then one has

$$\lfloor (d+1)(n-1)/2 \rfloor - I(a_{nd/2-k_j}) \geq (n-d-2)/2 + k_j \geq 0,$$

and if  $n/2 + k_j \leq d/2$ , since  $a_{nd/2-u} = a_{n(d/2+1)-d-1+u}$  by (2.3), one also has

$$\lfloor (d+1)(n-1)/2 \rfloor - I(a_{n(d/2+1)-d-1+k_j}) \geq (d-n)/2 - k_j \geq 0.$$

Similarly, one also has

$$\lfloor (d+1)(n-1)/2 \rfloor \geq I(a_{nd/2-k_{n(j)}}) \text{ or } \lfloor (d+1)(n-1)/2 \rfloor \geq I(a_{n(d/2+1)-d-1+k_{n(j)}}).$$

Therefore, by (2.5), we conclude that  $B(j, n(j)) \geq 0$  for each  $1 \leq j \leq \ell$ . In the same way, we also conclude that  $B(j, r) \geq 0$  for  $n(j+1) \leq r \leq n(j)$ .

Moreover, for  $n(t') \leq j \leq \ell$ , we define  $g_j$  by setting

$$g_j = \begin{cases} \left( \sum_{r=1}^{n(j+1)} h_{k_r} - \sum_{r=j+1}^{\ell} h_{k_r} \right) B(j, n(j+1)) + \sum_{r=n(j+1)+1}^{n(j)-1} h_{k_r} B(j, r) \\ \quad + \left( \sum_{r=j}^{\ell} h_{k_r} - \sum_{r=1}^{n(j)-1} h_{k_r} \right) B(j, n(j)), & \text{if } n(j+1) < n(j), \\ h_{k_j} B(j, n(j)), & \text{if } n(j+1) = n(j), \end{cases}$$

for  $n(t') + 1 \leq j \leq \ell$ , where we let  $n(\ell+1) = 0$ , and

$$g_{t'} = \begin{cases} \left( \sum_{r=1}^{n(t'+1)} h_{k_r} - \sum_{r=t'+1}^{\ell} h_{k_r} \right) B(t', n(t'+1)) + \sum_{r=n(t'+1)+1}^{n(t')-1} h_{k_r} B(t', r) \\ \quad + \left( \sum_{r=t'}^{\ell} h_{k_r} - \sum_{r=1}^{n(t')-1} h_{k_r} \right) B(t', t')/2, & \text{if } n(t'+1) < n(t'), \\ \left( \sum_{r=1}^{n(t')} h_r - \sum_{r=t'+1}^{\ell} h_r \right) B(t', t')/2, & \text{if } n(t'+1) = n(t') \end{cases}$$

when  $n(t') = t'$  and

$$g_{t'-1} = \left( \sum_{r=1}^{n(t')} h_r - \sum_{r=t'}^{\ell} h_r \right) B(t'-1, t'-1)/2$$

when  $n(t') = t' - 1$ . By definition of  $n(j)$  together with the nonnegativity of each  $B(j, r)$  for  $n(j+1) \leq r \leq n(j)$ , we obtain that  $g_j \geq 0$  for each  $j$ .

By using these notation, we can compute as follows:

$$\delta_i - \delta_{d+1-i} = \sum_{j=0}^{\ell} h_{k_j} a_{ni-k_j} - \sum_{j=0}^{\ell} h_{k_j} a_{n(d+1-i)-k_j} = B(0, 0)/2 + \sum_{j=n(t')}^{\ell} g_j.$$

Since  $B(0, 0) > 0$  and  $g_j \geq 0$  for each  $j$ , we have  $\delta_i - \delta_{d+1-i} > 0$ , as required.  $\square$

### 3. SEVERAL EXAMPLES OF $\delta$ -VECTORS CONCERNING UNIMODALITY

The goal of this section is to provide several kinds of  $\delta$ -vectors. Those concern unimodality, log-concavity and alternatingly increasingness.

**Remark 3.1.** (a) Let  $\mathcal{P} \subset \mathbb{R}^N$  be a lattice polytope and  $(\delta_0, \delta_1, \dots, \delta_d)$  its  $\delta$ -vector. If  $\mathcal{P}$  has IDP, then one has  $\delta_1^2 \geq \delta_0 \delta_2$ .

In fact, let  $\delta_1 = \ell$ . Then  $|\mathcal{P} \cap \mathbb{Z}^N| = \ell + d + 1$ . If  $\ell = 0$ , then we do not have to say anything from [13, Lemma 3.1]. Assume  $\ell > 0$ . From  $i(\mathcal{P}, m) = \sum_{i=0}^d \delta_i \binom{m+d-i}{d}$ , we also see that  $|2\mathcal{P} \cap \mathbb{Z}^N| = \binom{d+2}{2} + (d+1)\ell + \delta_2$ . Since  $\mathcal{P}$  has IDP, we have  $|m\mathcal{P} \cap \mathbb{Z}^N| \leq \binom{\ell+d+m}{m}$ . In particular,  $|2\mathcal{P} \cap \mathbb{Z}^N| \leq \binom{\ell+d+2}{2}$ . Hence  $\delta_2 \leq \binom{\ell+d+2}{2} - \binom{d+2}{2} - (d+1)\ell = (\ell^2 + \ell)/2$ . Therefore,  $\delta_1^2 - \delta_0 \delta_2 = \ell^2 - \delta_2 \geq \ell^2 - (\ell^2 + \ell)/2 = \ell(\ell - 1)/2$ . This is always nonnegative by  $\ell > 0$ , as required.

Moreover, one has  $\delta_2 \geq \delta_1$  by [10]. Note that  $\delta_1 \geq \delta_d$  always holds. (See Introduction.) Thus, we have  $\delta_2^2 \geq \delta_1 \delta_d$ . Hence, we obtain that all  $\delta$ -vectors of lattice polytopes of dimension at most 3 having IDP are always log-concave.

(b) The  $\delta$ -vectors of lattice polytopes of dimension at most 4 with at least one interior lattice point are always alternatingly increasing. In particular, it is unimodal. See [13, Proposition 3.2].

Before providing examples, we recall the well-known combinatorial technique how to compute the  $\delta$ -vector of a lattice simplex. Given a lattice simplex  $\mathcal{F} \subset \mathbb{R}^N$  of dimension  $d$  with the vertices  $v_0, v_1, \dots, v_d \in \mathbb{Z}^N$ , we set

$$\Lambda_{\mathcal{F}} = \left\{ \alpha \in \mathbb{Z}^{N+1} : \alpha = \sum_{i=0}^d r_i(v_i, 1), \ 0 \leq r_i < 1 \right\}.$$

We define the degree of  $\alpha = \sum_{i=0}^d r_i(v_i, 1) \in \Lambda_{\mathcal{F}}$  to be  $\deg(\alpha) = \sum_{i=0}^d r_i$ , i.e., the last coordinate of  $\alpha$ . Then we have the following:

**Lemma 3.2** (cf. [2, Corollary 3.11]). *Let  $\delta(\mathcal{F}) = (\delta_0, \delta_1, \dots, \delta_d)$ . Then, for each  $0 \leq i \leq d$ ,*

$$\delta_i = |\{\alpha \in \Lambda_{\mathcal{F}} : \deg(\alpha) = i\}|.$$

**3.1. Non-unimodal  $\delta$ -vectors.** First, we construct lattice polytopes which contain  $m$  interior lattice points whose  $\delta$ -vectors are not unimodal. By Remark 3.1 (b), if a lattice polytope has a non-unimodal  $\delta$ -vector, then its dimension is at least 5.

Let  $\mathbf{e}_1, \dots, \mathbf{e}_d$  be the unit coordinate vectors of  $\mathbb{R}^d$  and  $\mathbf{0}$  its origin.

**Example 3.3.** Let  $d \geq 5$  and  $m \geq 1$  be integers. Then there exists a lattice polytope of dimension  $d$  containing exactly  $m$  interior lattice points such that its  $\delta$ -vector is not unimodal.

The case  $d$  is odd: Let  $d = 2\ell + 1$ , where  $\ell \geq 2$ . We define  $\mathcal{P}_{\text{odd}}(\ell, m)$  by setting the convex hull of  $v_0, v_1, \dots, v_d$ , where  $M = 2(2m + 1)(\ell + 1)$  and

$$\begin{aligned} v_i &= \begin{cases} \mathbf{0}, & i = 0, \\ \mathbf{e}_i, & i = 1, \dots, d-1, \end{cases} \\ v_d &= (M - 2(\ell + 1)m)\mathbf{e}_1 + (M - 1)(\mathbf{e}_2 + \dots + \mathbf{e}_{d-1}) + M\mathbf{e}_d. \end{aligned}$$

Let  $\delta(\mathcal{P}_{\text{odd}}(\ell, m)) = (\delta_0, \delta_1, \dots, \delta_d)$ . Then we can calculate from Lemma 3.2 that

$$\delta_i = \left| \left\{ j \in \mathbb{Z} : \left\lceil \frac{2\ell j}{M} + \left\{ \frac{2(\ell+1)mj}{M} \right\} \right\rceil = i, 0 \leq j \leq M-1 \right\} \right|,$$

where  $\{r\}$  denotes the fraction part of a rational number  $r$ , i.e.,  $\{r\} = r - \lfloor r \rfloor$ . Let  $f(j) = \left\lceil \frac{2\ell j}{M} + \left\{ \frac{2(\ell+1)mj}{M} \right\} \right\rceil = \left\lceil \frac{\ell j}{(2m+1)(\ell+1)} + \left\{ \frac{mj}{2m+1} \right\} \right\rceil$ .

- Let  $j = (2m+1)k + 2p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2$ . Then

$$f(j) = \left\lceil \frac{\ell k}{\ell+1} + \frac{rm}{2m+1} + \frac{(\ell-1)p + \ell r}{(2m+1)(\ell+1)} \right\rceil.$$

- Let  $j = (2m+1)k$ , where  $1 \leq k \leq 2\ell+1$ . Then

$$f(j) = \left\lceil \frac{\ell k}{\ell+1} \right\rceil.$$

(i) We prove that  $\mathcal{P}_{\text{odd}}(\ell, m)$  contains exactly  $m$  lattice points in its interior, i.e., we may check  $\delta_d = \delta_{2\ell+1} = m$ .

- (a) For  $j = (2m+1)k + 2p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2$ , if  $k \leq 2\ell$ , then we see that

$$\begin{aligned} f(j) &= \left\lceil \frac{\ell k}{\ell+1} + \frac{rm}{2m+1} + \frac{p(\ell-1) + \ell r}{(2m+1)(\ell+1)} \right\rceil \\ &\leq \left\lceil \frac{2\ell^2}{\ell+1} + \frac{2m}{2m+1} + \frac{(m-1)(\ell-1) + 2\ell}{(2m+1)(\ell+1)} \right\rceil \\ &= 2\ell - 1 + \left\lceil \frac{m\ell + 3m + 2}{(2m+1)(\ell+1)} \right\rceil \leq 2\ell. \end{aligned}$$

Thus, if  $f(j) = 2\ell+1$ , then  $k = 2\ell+1$ . Similarly, if  $r = 1$ , then we see that  $f(j) \leq 2\ell$ . Thus, if  $f(j) = 2\ell+1$ , then  $r = 2$ . On the other hand, when  $k = 2\ell+1$  and  $r = 2$ , we obtain that

$$f(j) = 2\ell + \left\lceil \frac{p(\ell-1) + \ell + 2m}{(2m+1)(\ell+1)} \right\rceil = 2\ell + 1 \quad \text{for each } 0 \leq p \leq m-1.$$

- (b) For  $j = (2m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , we see that  $f(j) \leq 2\ell$ .

From the above (a) and (b), we conclude that  $\delta_{2\ell+1} = \delta_d = m$ .

- (ii) We prove the non-unimodality of  $(\delta_0, \dots, \delta_d)$ .

- (a) The following statements imply that  $\delta_1 \leq m+1$ .

- For  $j = (2m+1)k + 2p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2$ , if  $k \geq 1$ , then we have  $f(j) \geq 2$ . Similarly, if  $r = 2$ , then  $f(j) \geq 2$ . Thus, if  $f(j) = 1$ , then  $k = 0$  and  $r = 1$ .
- Moreover, for  $j = (2m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , we see that  $f(j) = 1$  if and only if  $j = 1$ .

- (b) The following imply that  $\delta_\ell \geq 2m+2$ .

- For  $j = (2m+1)k + 2p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2$ , if  $k = \ell$  and  $r = 1$ , then we see that  $f(j) = \ell$ . Similarly, if  $k = \ell-1$  and  $r = 2$ , then we see that  $f(j) = \ell$ .

- Moreover, for  $j = (2m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , one has  $f(j) = \ell$  if and only if  $k = \ell$  or  $k = \ell+1$ .
- (c) The following imply that  $\delta_{\ell+1} \leq 2m+1$ .
  - For  $j = (2m+1)k+2p+r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2$ , if  $k \leq \ell-1$ , then  $f(j) \leq \ell$ . Moreover,  $k \geq \ell+2$ , then  $f(j) \geq \ell+2$ . In addition, if  $k = \ell+1$  and  $r = 2$ , then  $f(j) \geq \ell+2$ . Furthermore, if  $k = \ell$  and  $r = 1$ , then  $f(j) \leq \ell$ . Thus, it must be satisfied that  $(k, r) = (\ell, 2)$  or  $(k, r) = (\ell+1, 1)$  when  $f(j) = \ell+1$ .
  - Moreover, for  $j = (2m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , one has  $f(j) = \ell+1$  if and only if  $k = \ell+2$ .
- (d) The following imply that  $\sum_{i=\ell+2}^{2\ell} \delta_i \geq 2m\ell+\ell-1$ . Then we notice that  $2m\ell+\ell-1 = (\ell-1)(2m+1) + 2m$ . Hence, we obtain that  $\max\{\delta_{\ell+2}, \dots, \delta_{2\ell}\} \geq 2m+2$ .
  - For  $j = (2m+1)k+2p+r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2$ , if  $\ell+2 \leq k \leq 2\ell$ , then  $\ell+2 \leq f(j) \leq 2\ell$ . Moreover, if  $k = 2\ell+1$  and  $r = 1$ , then  $f(j) = 2\ell$ . In addition, if  $k = \ell+1$  and  $r = 2$ , then  $f(j) = \ell+2$ .
  - Moreover, for  $j = (2m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , one has  $\ell+2 \leq f(j) \leq 2\ell$  if and only if  $\ell+3 \leq k \leq 2\ell+1$ .

Summarizing the above (a)–(d), one sees that

$$\delta_1 \leq m+1, \delta_\ell \geq 2m+2, \delta_{\ell+1} \leq 2m+1 \text{ and } \max\{\delta_{\ell+2}, \dots, \delta_{2\ell}\} \geq 2m+2.$$

Hence,

$$\delta_1 < \delta_\ell > \delta_{\ell+1} < \max\{\delta_{\ell+2}, \dots, \delta_{2\ell}\}.$$

This shows the non-unimodality of  $(\delta_0, \delta_1, \dots, \delta_{2\ell+1})$ .

The case  $d$  be even: Let  $d = 2\ell+2$ , where  $\ell \geq 2$ . We define  $\mathcal{P}_{\text{even}}(\ell, m)$  by setting the convex hull of  $v_0, v_1, \dots, v_d$ , where  $M = 2(3m+1)(\ell+1)$  and

$$v_i = \begin{cases} \mathbf{0}, & i = 0, \\ \mathbf{e}_i, & i = 1, \dots, d-1, \end{cases}$$

$$v_d = (M - 2(\ell+1)m)(\mathbf{e}_1 + \mathbf{e}_2) + (M-1)(\mathbf{e}_3 + \dots + \mathbf{e}_{d-1}) + M\mathbf{e}_d.$$

Let  $\delta(\mathcal{P}_{\text{even}}(\ell, m)) = (\delta_0, \delta_1, \dots, \delta_d)$ . Then we can calculate from Lemma 3.2 that

$$\delta_i = \left| \left\{ j \in \mathbb{Z} : \left\lceil \frac{2\ell j}{M} + 2 \left\{ \frac{2(\ell+1)mj}{M} \right\} \right\rceil = i, 0 \leq j \leq M-1 \right\} \right|.$$

$$\text{Let } g(j) = \left\lceil \frac{2\ell j}{M} + 2 \left\{ \frac{2(\ell+1)mj}{M} \right\} \right\rceil = \left\lceil \frac{\ell j}{(3m+1)(\ell+1)} + 2 \left\{ \frac{mj}{3m+1} \right\} \right\rceil.$$

- Let  $j = (3m+1)k+3p+r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2, 3$ . Then

$$g(j) = \left\lceil \frac{\ell k}{\ell+1} + \frac{2rm}{3m+1} + \frac{(\ell-2)p + \ell r}{(3m+1)(\ell+1)} \right\rceil.$$

- Let  $j = (3m+1)k$ , where  $1 \leq k \leq 2\ell+1$ . Then

$$g(j) = \left\lceil \frac{\ell k}{\ell+1} \right\rceil.$$

(i) We prove that  $\mathcal{P}_{\text{even}}(\ell, m)$  contains exactly  $m$  lattice points in its interior, i.e., we may check  $\delta_d = \delta_{2\ell+2} = m$ .

- (a) For  $j = (2m+1)k + 3p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2, 3$ , if  $k \leq 2\ell$ , then we see that  $g(j) \leq 2\ell+1$ . Thus, if  $g(j) = 2\ell+2$ , then  $k = 2\ell+1$ . Similarly, if  $r \leq 2$ , then we see that  $f(j) \leq 2\ell+1$ . Thus, if  $g(j) = 2\ell+2$ , then  $r = 3$ . On the other hand, when  $k = 2\ell+1$  and  $r = 3$ , we obtain that

$$g(j) = 2\ell+1 + \left\lceil \frac{p(\ell-2) + \ell + 3m - 1}{(3m+1)(\ell+1)} \right\rceil = 2\ell+2 \quad \text{for } 0 \leq p \leq m-1.$$

- (b) For  $j = (3m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , we see that  $g(j) \leq 2\ell+1$ .

From the above (a) and (b), we conclude that  $\delta_{2\ell+2} = \delta_d = m$ .

- (ii) We prove the non-unimodality of  $(\delta_0, \dots, \delta_d)$ .

- (a) The following statements imply that  $\delta_1 \leq m+1$ .
- For  $j = (3m+1)k + 3p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2, 3$ , if  $k \geq 1$ , then we have  $g(j) \geq 2$ . Similarly, if  $r \geq 2$ , then  $g(j) \geq 2$ . Thus, if  $f(j) = 1$ , then  $k = 0$  and  $r = 1$ .
  - Moreover, for  $j = (3m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , we see that  $g(j) = 1$  if and only if  $j = 1$ .
- (b) The following imply that  $\delta_\ell \geq 3m+2$ .
- For  $j = (3m+1)k + 3p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2, 3$ , if  $(k, r) = (\ell-2, 3), (\ell-1, 2)$  or  $(\ell, 1)$ , then we see that  $g(j) = \ell$ .
  - Moreover, for  $j = (3m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , one has  $g(j) = \ell$  if and only if  $k = \ell$  or  $k = \ell+1$ .
- (c) The following imply that  $\delta_{\ell+1} \leq 3m+1$ .
- For  $j = (3m+1)k + 3p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2, 3$ , we see that  $g(j) = \ell+1$  only if  $(k, r) = (\ell-1, 3), (\ell, 2)$  or  $(\ell+1, 1)$ .
  - Moreover, for  $j = (3m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , one has  $g(j) = \ell+1$  if and only if  $k = \ell+2$ .
- (d) The following imply that  $\sum_{i=\ell+2}^{2\ell} \delta_i \geq 3m\ell + \ell - 1$ . Then we notice that  $3m\ell + \ell - 1 = (\ell-1)(3m+1) + 3m$ . Hence, we obtain that  $\max\{\delta_{\ell+2}, \dots, \delta_{2\ell}\} \geq 3m+2$ .
- For  $j = (3m+1)k + 3p + r$ , where  $0 \leq k \leq 2\ell+1$ ,  $0 \leq p \leq m-1$  and  $r = 1, 2, 3$ , if  $\ell+2 \leq k \leq 2\ell-1$ , then  $\ell+2 \leq g(j) \leq 2\ell$ . Moreover, if  $(k, r) = (\ell, 3), (\ell+1, 2), (\ell+1, 3), (2\ell, 1), (2\ell, 2), (2\ell+1, 1)$ , then  $\ell+2 \leq g(j) \leq 2\ell$ .
  - Moreover, for  $j = (3m+1)k$ , where  $1 \leq k \leq 2\ell+1$ , one has  $\ell+2 \leq g(j) \leq 2\ell$  if and only if  $\ell+3 \leq k \leq 2\ell+1$ .

Summarizing the above (a)–(d), one sees that

$$\delta_1 \leq m+1, \quad \delta_\ell \geq 3m+2, \quad \delta_{\ell+1} \leq 3m+1 \quad \text{and} \quad \max\{\delta_{\ell+2}, \dots, \delta_{2\ell}\} \geq 3m+2.$$

Hence,

$$\delta_1 < \delta_\ell > \delta_{\ell+1} < \max\{\delta_{\ell+2}, \dots, \delta_{2\ell}\},$$

as desired.

### 3.2. Unimodal but neither log-concave nor alternatingly increasing $\delta$ -vectors.

Next, we give examples of lattice polytopes whose  $\delta$ -vectors are unimodal but neither log-concave nor alternatingly increasing for odd dimensions.

**Example 3.4.** Let  $d \geq 5$  be an odd number and  $m \geq 1$  an integer. We define  $\mathcal{P}(d, m)$  by setting the convex hull of  $v_0, v_1, \dots, v_d$ , where  $M = 2(d-1)m + 2$  and

$$v_i = \begin{cases} \mathbf{0}, & i = 0, \\ \mathbf{e}_i, & i = 1, \dots, d-1, \\ (M-d+1)\mathbf{e}_1 + (M-1)(\mathbf{e}_2 + \dots + \mathbf{e}_{d-1}) + M\mathbf{e}_d, & i = d. \end{cases}$$

Then it can be computed that  $\delta(\mathcal{P}(d, m)) = (\delta_0, \delta_1, \dots, \delta_d)$  is equal to

$$\delta_i = \left| \left\{ j \in \mathbb{Z} : \left\lceil \frac{(d-1)j}{2(d-1)m+2} + \left\{ \frac{(d-1)j}{2(d-1)m+2} \right\} \right\rceil = i, 0 \leq j \leq M-1 \right\} \right|.$$

Let  $f(j) = \left\lceil \frac{(d-1)j}{2(d-1)m+2} + \left\{ \frac{(d-1)j}{2(d-1)m+2} \right\} \right\rceil$ . For  $1 \leq j \leq (d-1)m+1$ , we see the following:

- One has  $f(j) = 1$  if  $1 \leq j \leq m$ ;
- For  $2 \leq k \leq (d-1)/2$ , we have  $f(j) = k$  if  $(2k-3)m+1 \leq j \leq (2k-1)m$ ;
- One has  $f(j) = (d+1)/2$  if  $(d-2)m+1 \leq j \leq (d-1)m$ ;
- One has  $f((d-1)m+1) = (d-1)/2$ .

For  $(d-1)m+2 \leq j \leq 2(d-1)m+1$ , it is easy that  $f(j) = (d-1)/2 + f(j - (d-1)m - 1)$ . Therefore, we conclude that

$$\delta(\mathcal{P}(d, m)) = (1, m, 2m, \dots, 2m, \underbrace{2m+1}_{\delta_{(d-1)/2}}, 2m, \dots, 2m, m).$$

Since  $\delta_{(d-1)/2} > \delta_{(d+1)/2}$  and  $\delta_{(d-1)/2}\delta_{(d+3)/2} > \delta_{(d+1)/2}^2$ , this  $\delta$ -vector is neither log-concave nor alternatingly increasing. On the other hand, this  $\delta$ -vector is unimodal.

**3.3. Alternatingly increasing but not log-concave  $\delta$ -vectors.** Next, we give examples of lattice polytopes whose  $\delta$ -vectors are alternatingly increasing but not log-concave.

**Example 3.5.** Let  $d \geq 4$  and  $m \geq 1$  be integers. We define  $\mathcal{P}(d, m)$  by setting the convex hull of  $v_0, v_1, \dots, v_d$ , where  $M = (\lceil (d+1)/2 \rceil m + 1) \lceil (d+2)/2 \rceil$  and

$$v_i = \begin{cases} \mathbf{0}, & i = 0, \\ \mathbf{e}_i, & i = 1, \dots, d-1, \end{cases}$$

$$v_d = (M - \lceil (d+2)/2 \rceil m)(\mathbf{e}_1 + \dots + \mathbf{e}_{\lfloor d/2 \rfloor}) + (M-1)(\mathbf{e}_{\lfloor d/2 \rfloor + 1} + \dots + \mathbf{e}_{d-1}) + M\mathbf{e}_d.$$

The case  $d$  is odd: Let  $d' = (d+1)/2$ . Then  $M = (d'm + 1)(d' + 1)$ . It can be computed that  $\delta(\mathcal{P}(d, m)) = (\delta_0, \delta_1, \dots, \delta_d)$  is equal to

$$\delta_i = \left| \left\{ j \in \mathbb{Z} : \left\lceil \frac{d'j}{(d'm+1)(d'+1)} + \left\{ \frac{m j}{d'm+1} \right\} (d'-1) \right\rceil = i, 0 \leq j \leq M-1 \right\} \right|.$$

Let  $f(j) = \left\lceil \frac{d'j}{(d'm+1)(d'+1)} + \left\{ \frac{m j}{d'm+1} \right\} (d'-1) \right\rceil$ .

For each  $j = 1, \dots, M - 1$ , we have a unique expression such that  $\ell(d'm + 1)$  for some  $1 \leq \ell \leq d'$  or  $j = p(d'm + 1) + qd' + r$ , where  $0 \leq p \leq d'$ ,  $0 \leq q \leq m - 1$  and  $1 \leq r \leq d'$ . Thus

$$f(j) = \ell - 1 + \left\lceil \frac{d' + 1 - \ell}{d' + 1} \right\rceil = \ell \text{ if } j = \ell(d'm + 1), \text{ and}$$

$$f(j) = p + r - 1 + \left\lceil \frac{d' + 1 - p - r}{d' + 1} - \frac{mr - q}{(d'm + 1)(d' + 1)} \right\rceil \text{ if } j = p(d'm + 1) + qd' + r.$$

Note that  $1 \leq mr - q \leq d'm$ . Hence we obtain that

$$\begin{aligned} f(j) = i \text{ for } 1 \leq i \leq d' - 1 &\iff \ell = i \text{ or } p + r = i, \\ f(j) = d' &\iff \ell = d' \text{ or } p + r = d' \text{ or } p + r = d' + 1, \\ f(j) = i \text{ for } d' + 1 \leq i \leq d &\iff p + r = i + 1. \end{aligned}$$

From these observations, we conclude that

$$\delta(\mathcal{P}(d, m)) = (1, m + 1, 2m + 1, \dots, (d' - 1)m + 1, \underbrace{2d'm + 1}_{\delta_{d'}}, (d' - 1)m, \dots, m).$$

Clearly, this is alternatingly increasing, while this is not log-concave by  $\delta_{d'}\delta_{d'+2} > \delta_{d'+1}^2$ .

The case  $d$  is even: Let  $d' = d/2$ . Then  $M = (d'm + m + 1)(d' + 1)$ . It can be computed that  $\delta(\mathcal{P}(d, m)) = (\delta_0, \delta_1, \dots, \delta_d)$  is equal to

$$\delta_i = \left| \left\{ j \in \mathbb{Z} : \left\lceil \frac{d'j}{(d'm + m + 1)(d' + 1)} + \left\{ \frac{mj}{d'm + m + 1} \right\} d' \right\rceil = i, 0 \leq j \leq M - 1 \right\} \right|.$$

Let  $f(j) = \left\lceil \frac{d'j}{(d'm + m + 1)(d' + 1)} + \left\{ \frac{mj}{d'm + m + 1} \right\} d' \right\rceil$ .

For each  $1 \leq j \leq M - 1$ , we have a unique expression such that  $\ell(d'm + m + 1)$  for some  $1 \leq \ell \leq d'$  or  $j = p(d'm + m + 1) + q(d' + 1) + r$ , where  $0 \leq p \leq d'$ ,  $0 \leq q \leq m - 1$  and  $1 \leq r \leq d' + 1$ . Thus

$$f(j) = \ell - 1 + \left\lceil \frac{d' + 1 - \ell}{d' + 1} \right\rceil = \ell \text{ if } j = \ell(d'm + m + 1),$$

$$f(j) = p + r - 1 + \left\lceil \frac{d' + 1 - p - r}{d' + 1} \right\rceil \text{ if } j = p(d'm + m + 1) + q(d' + 1) + r.$$

Hence we obtain that

$$\begin{aligned} f(j) = i \text{ for } 1 \leq i \leq d' - 1 &\iff \ell = i \text{ or } p + r = i, \\ f(j) = d' &\iff \ell = d' \text{ or } p + r = d' \text{ or } p + r = d' + 1, \\ f(j) = i \text{ for } d' + 1 \leq i \leq d &\iff p + r = i + 1. \end{aligned}$$

From these observations, we conclude that

$$\delta(\mathcal{P}(d, m)) = (1, m + 1, 2m + 1, \dots, (d' - 1)m + 1, \underbrace{(2d' + 1)m + 1}_{\delta_{d'}}, d'm, (d' - 1)m, \dots, m).$$

Clearly, this is alternatingly increasing, while this is not log-concave by  $\delta_{d'}\delta_{d'+2} > \delta_{d'+1}^2$ .

For the case of lattice polytopes of dimension 3, we note the following:

**Remark 3.6.** Let  $(\delta_0, \delta_1, \delta_2, \delta_3)$  be the  $\delta$ -vector of some lattice polytope of dimension 3 with  $\delta_3 \neq 0$ . Since  $\delta_2 \geq \delta_1 \geq \delta_3$ , we always have  $\delta_2^2 \geq \delta_1 \delta_3$ . Moreover, as mentioned in Remark 3.1 (b),  $(\delta_0, \delta_1, \delta_2, \delta_3)$  is always alternately increasing. Thus, if  $(\delta_0, \delta_1, \delta_2, \delta_3)$  is alternately increasing but not log-concave, then it should be  $\delta_2 > \delta_1^2$ . On the other hand, such a lattice polytope never has IDP by Remark 3.1 (a).

For example, the  $\delta$ -vector of the lattice polytope with its vertices  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, 2(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$  is equal to  $(1, 1, 2, 1)$ . This is not log-concave.

**3.4. Log-concave but not alternately increasing  $\delta$ -vectors.** Finally, we supply a couple of examples of lattice polytopes whose  $\delta$ -vectors are log-concave but not alternately increasing in low dimensions.

Let  $\mathcal{P}_3 \subset \mathbb{R}^6$  be a lattice polytope of dimension 6 whose vertices are

$$\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_4, 2(\mathbf{e}_1 + \dots + \mathbf{e}_4) + 3\mathbf{e}_5, 16(\mathbf{e}_1 + \dots + \mathbf{e}_4) + 3\mathbf{e}_5 + 30\mathbf{e}_6.$$

Then we have  $\delta(\mathcal{P}_3) = (1, 6, 20, 22, 23, 15, 3)$ . Moreover, let  $\mathcal{P}_4$  be a lattice polytope whose vertices are

$$\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_4, 2(\mathbf{e}_1 + \dots + \mathbf{e}_4) + 3\mathbf{e}_5, 22(\mathbf{e}_1 + \dots + \mathbf{e}_4) + 3\mathbf{e}_5 + 42\mathbf{e}_6.$$

Then we have  $\delta(\mathcal{P}_4) = (1, 7, 28, 31, 32, 23, 4)$ . Both of them are log-concave but not alternately increasing.

Similarly, we have checked the existence of some more lattice polytopes of dimension 6 whose  $\delta$ -vectors are log-concave but not alternately increasing.

**3.5. Future works.** We remain the following problems:

- Problem 3.7.** *If there exists, construct a family of lattice polytopes whose  $\delta$ -vectors are*
- (a) *unimodal but neither log-concave nor alternately increasing for even dimensions;*
  - (b) *alternately increasing but not log-concave for dimension 3;*
  - (c) *log-concave but not alternately increasing for dimension at least 5.*

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